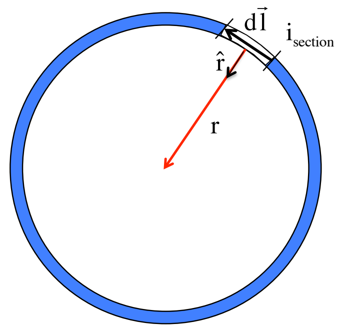
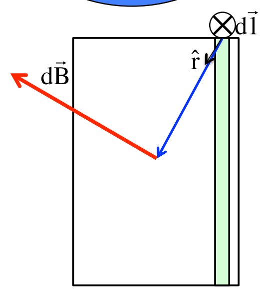
Problem 30.4

Define a strip of the induced current as $i_{section}$. From the side, a tiny section "dl" of that current would set up a magnetic field at the center of the cylinder defined by Biot Savart as:

$$dB = \frac{\mu_o i_{section}}{4\pi}\frac{d\vec{l}x\hat{r}}{r^2}$$

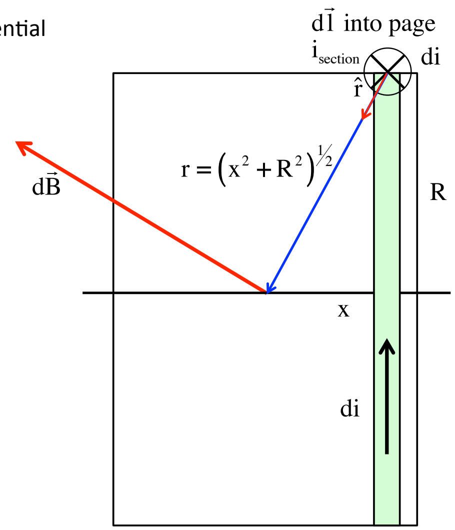


How this gets tricky can be seen by looking at the cylinder from the side. If the section associated with $d\vec{l}$ is right at the top of the page going into the page for the swath of current viewed, the right-thumb rule show that the direction of dB is seen not directed down the axis.



According to Biot-Savart, that differential magnetic field vector will equal:

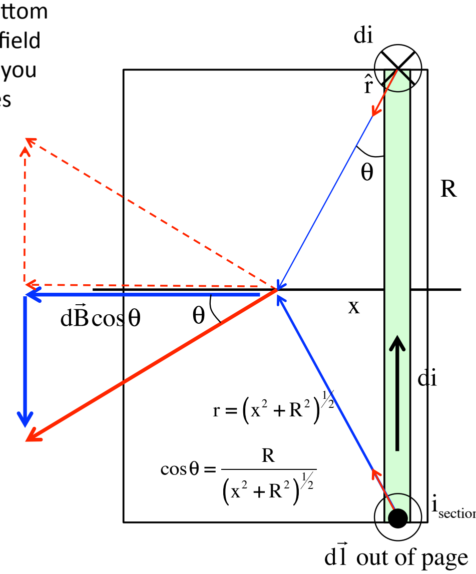
$$dB = \frac{\mu_o\left(i_{sect}\right)}{4\pi r^2}d\vec{l}x\hat{r}$$
$$= \frac{\mu_o\left(i_{sect}\right)}{4\pi r^2}dl\sin 90^\circ \quad (\sin 90^\circ \to 1)$$
$$= \frac{\mu_o\left(i_{sect}\right)}{4\pi r^2}dl$$
$$= \frac{\mu_o\left(i_{sect}\right)dl}{4\pi\left[\left(x^2+R^2\right)^{1/2}\right]^2}$$
$$= \frac{\mu_o\left(i_{sect}\right)dl}{4\pi\left(x^2+R^2\right)}$$



Look at the current section at the bottom of the swath. The vertical magnetic field components will add to zero leaving you with differential magnetic field pieces in the x-direction only. That means that the net field will be along the central axis of the hoop.

For our differential piece, we can write:

$$dB_x = dB\cos\theta$$
$$= \frac{\mu_o\left(i_{section}\right)(dl)}{4\pi\left(x^2+R^2\right)}\cos\theta$$
$$= \frac{\mu_o\left(i_{section}\right)(dl)}{4\pi\left(x^2+R^2\right)}\frac{R}{\left(x^2+R^2\right)^{1/2}}$$
$$= \frac{\mu_o R\left(i_{section}\right)(dl)}{4\pi\left(x^2+R^2\right)^{3/2}}$$



If the width of the cylinder is L, the current density in amps per meter is such that

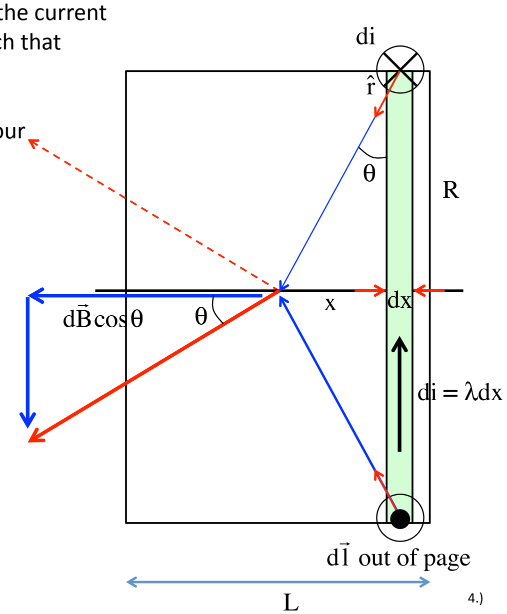
$$i_{total} = \lambda L \quad \Rightarrow \quad \lambda = i_{total}/L$$

With that, the current through our swath of width "dx" becomes:

$$i_{section} = \lambda dx$$

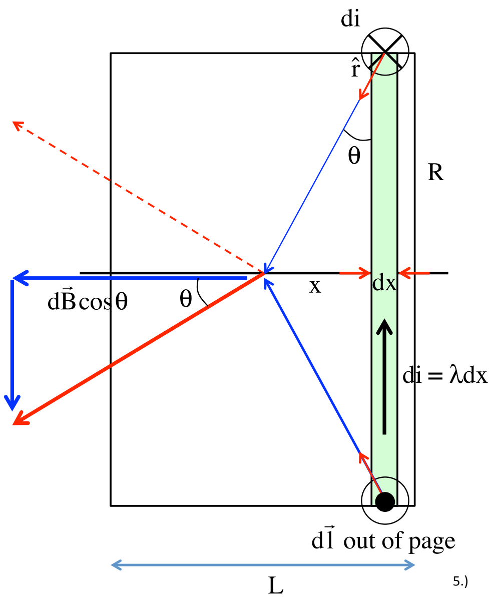
and

$$dB_x = \frac{\mu_o R\left(i_{section}\right)dl}{4\pi\left(x^2+R^2\right)^{3/2}}$$
$$= \frac{\mu_o R\left(\lambda dx\right)dl}{4\pi\left(x^2+R^2\right)^{3/2}}$$



To get the total differential magnetic field due to the entire swath, we integrate over "dl" and get:

$$dB_{x,total} = \frac{\mu_o R(\lambda dx)}{4\pi(x^2+R^2)^{3/2}}\oint dl$$
$$= \frac{\mu_o R(\lambda dx)}{4\pi(x^2+R^2)^{3/2}}(2\pi R)$$
$$= \frac{\mu_o R^2(\lambda dx)}{2(x^2+R^2)^{3/2}}$$

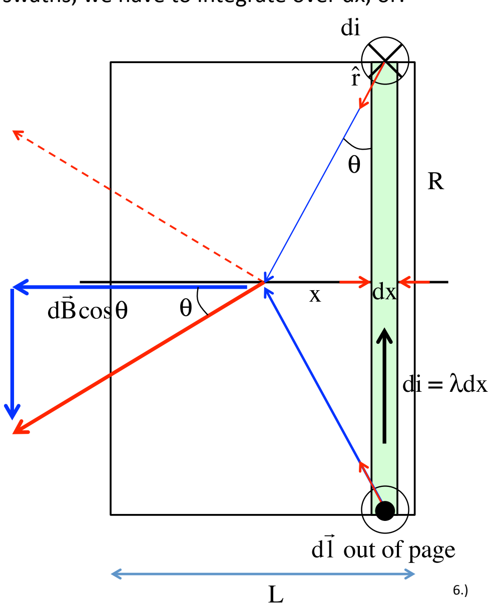


To get B due to all the differential swaths, we have to integrate over dx, or:

$$B = 2\int_{x=0}^{L/2} dB_{x,total}$$
$$= 2\left(\int_{x=0}^{L/2} \frac{\mu_o R^2(\lambda dx)}{2(x^2+R^2)^{3/2}}\right)$$
$$= (2)\left(\frac{\mu_o R^2(\lambda)}{2}\right)\int_{x=0}^{L/2}\frac{dx}{(x^2+R^2)^{3/2}}$$
$$= \mu_o R^2(\lambda)\left[\frac{x}{R^2(x^2+R^2)^{1/2}}\right]\Bigg|_{x=0}^{L/2}$$

Substituting in

$$\lambda = i_{total}/L$$

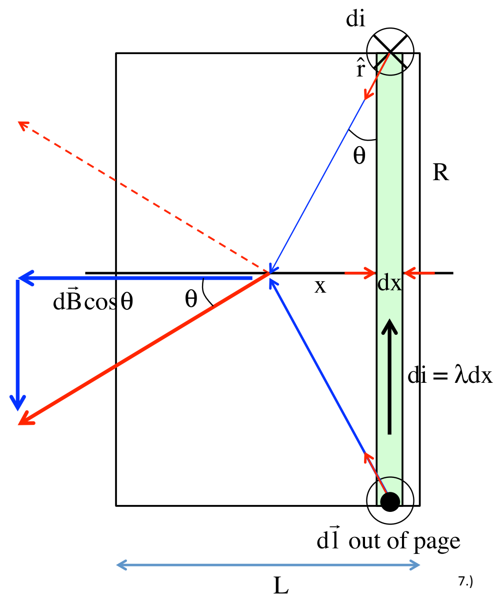


We get:

$$= \mu_o R^2(\lambda)\left[\frac{x}{R^2(x^2+R^2)^{1/2}}\right]\Bigg|_{x=0}^{L/2}$$
$$= \mu_o R^2\left(i_{total}/L\right)\left[\frac{L/2}{R^2\left((L/2)^2+R^2\right)^{1/2}}\right]$$
$$= \frac{\mu_o i_{total}}{2}\left[\frac{1}{\left((L/2)^2+R^2\right)^{1/2}}\right]$$

(And kindly note that if L goes to zero, as would be the case if the cylinder was a hoop, we get the expected):

$$B = \frac{\mu_o i_{total}}{2R}$$



In any case, with our derived expression, the magnetic field at the center of the cylinder becomes:

$$B = \frac{\mu_o i}{2R}$$
$$= \frac{\mu_o\left(\frac{ev}{2\pi R}\right)}{2R}$$
$$= \frac{\mu_o ev}{4\pi R^2}$$
$$= \frac{(4\pi x10^{-7}\ T\bullet m/A)(1.6x10^{-19}\ C)(2.19x10^{6}\ m/s)}{4\pi(5.29x10^{-11}\ m)^2}$$
$$= 12.5\ T$$